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# Fractals, their transfer matrices and their eigen-dimensional sequences

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**Abstract.** This paper defines and studies a general algorithm for constructing new families of fractals in Euclidean space. This algorithm involves a sequences of linear interscale transformations that proceed from large to small scales. We find that the fractals obtained in this fashion decompose *in intrinsic fashion* into linear combinations of a variable number of "addend' fractals. The addends' relative weights and fractal dimensionalities are obtained explicitly through an interscale matrix, which we call the *transfer matrix of the fractal* (TMF). We first demonstrate by a series of examples, then prove rigorously, that the eigenvalues of our TMFs are real and positive, and that the fractal dimensions of the addend fractals are the logarithms of the eigenvalues of our TMF. We say that these dimensions form the overall fractal's *eigendimensional sequence*. The eigenvalues of our TMF are integers in the non-random variants of the construction, but are non-integer in the random variants. A geometrical interpretation of the eigenvalues and the eigenvectors is given. Our TMF have other striking and very special properties that deserve additional attention.

## 1. Introduction

Scaling fractals, (Mandelbrot 1975, 1977, 1982) which are self-similar geometrical figures that go beyond the standard figures in Euclid, are natural models for physical systems in which some properties are invariant under scale transformation. In particular, statistical mechanical systems that undergo high-order phase transitions are self-similar below the correlation length $\xi$, which is their only characteristic length scale. Since the geometry underlying these systems is surely made of neither lines nor planes, it must involve self-similar fractals. Statistical mechanics has a very efficient method for studying such systems, the renormalisation group: it moves from the microscopic length scales towards increasingly larger scales up to $\xi$ (Fisher 1974, Aharony 1976) by multiplying lengths by a 'base' $b$. In a particular type of second-order phase transition, namely the percolation problem, (Stauffer 1979) fractals have become very important (Gefen *et al* 1981b, Mandelbrot 1984, Mandelbrot and Given 1984, Aharony 1984, Proc. Gaithersburg Conf. 1984). The application of renormalisation

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group techniques for physical systems constructed upon fractals was also studied in detail (Gefen *et al* 1980, 1981a, 1983). Recently, the applications of fractals in physics have grown explosively, (Mandelbrot 1984, Mandelbrot and Given 1984, Gefen 1983) and it is no longer possible to summarise them in a few lines. These and related developments create a strong continuing need for new fractal constructions. The present paper proposes such a construction. Many others can be found in the recent literature, such as the fractal squigs†. The original phase of factal geometry and of its application to physics had been characterised by the 'adoption' of various scattered shapes that mathematicians had previously advanced for totally different purposes, e.g., Sierpiński gaskets and carpets. This phase is now over and fractal geometry and its applications are mostly concerned with new fractal constructions.

A conspicuous feature of the fractals introduced in this paper is the need to generalise the definition of fractal dimensionality as a similarity exponent. In the simplest cases, $D$ is known to be the (unique) solution of the 'dimension generating equation' $Nb^{-D}=1$, or of various generalisations, such as the randomised variant $\langle N\rangle b^{-D}=1$. When different parts of the fractal are deduced from the whole by different ratios $r_j=1/b_j$, the dimension generating equation generalises further to become $\Sigma b_j^{-D}=1$. In this paper, the dimension generating equation generalises again, to take the form $\|\langle N_{ij}\rangle - b^D\delta_{ij}\|=0$, where the quantities $N_{ij}$ form a matrix that we shall call the 'interscale transfer matrix of the fractal' (TMF). More precisely, our first finding is that $D=\log_b\lambda_1$, where $\lambda_1$ is the leading eigenvalue of the TMF. But in addition, we find that the other eigenvalues $\lambda_i$ of the TMF are real and positive, that in the non-random case they are integers, and that our fractal includes intrinsically a collection of other fractals, each of them endowed with a fractal dimensionality of the form $\log_b\lambda_i$. As a result, the unique dimensionality obtained as the solution of $\langle N\rangle b^D=1$ is replaced by a number of 'eigen-dimensionalities' that have a clear geometric and physical meaning.

### *1.1. Summary*

Matrix generalisations of the dimension-generating equation $Nb^{-D}=1$ occur in other contexts‡, but out first encounter with a matrix whose eigenvalues are real and positive was a surprise. We propose to explain this observation, first by examples then by rigorous argument.

Thus, this paper divides into two sections having sharply different styles. Section 2 describes our algorithm, in a somewhat discursive fashion meant to create intuition. We start by an example in the plane, and proceed to diverse generalisations, one of them in a higher space. The construction of the TMF is explained and observations concerning its mathematical features are discussed. Section 3 presents some mathematical results, namely diverse theorems regarding the eigenvalues and eigenvectors of the TMFs. We realise that physicists will tend to find the treatment in this section to be excessively concise, but we feel that it is important to put this treatment in the record.

† Mandelbrot (1982 chapter 24), Mandelbrot (1984), Mandelbrot and Given (1984) and Proc. Gaithersburg Conf. (1984). See also Mandelbrot (1978a and b, figures 5 and 6) also Peyrière (1978, 1981) and Mandelbrot and Given (1984).

‡ A point of history is of interest to one of the authors (BBM). The study of word frequencies involves an exponent that has lately been reinterpreted as a factal dimensionality $D$ (Mandelbrot 1982, page 346). A very early reference is Mandelbrot (1955) where $D$ is obtained as the logarithm of the leading eigenvalue of a matrix that is in effect a TMF.

Section 4 points out open questions, and sketches possible applications of the TMF in physics.

### *1.2. Background in an elementary fractal construction*

The nature of our construction can be illustrated by examples of sets on the line. First example (Mandelbrot 1982, page 151): it is often asserted that the closed segment [0, 1] is self-similar, but this is not quite the case, since the sum of the closed segments $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ counts the point $\frac{1}{2}$ twice: thus the sum of the parts is the segment [0, 1] *plus* the point $\frac{1}{2}$. To achieve full self-similarity, one must delete the endpoint 1 and consider the half-open segment, written as [0, 1[. Alternatively, [0, 1] can be viewed as the sum of two self-similar sets: the segment [0, 1[ ($D = 1$) and the point 1 ($D = 0$).

Second example: as is well known, the triadic Cantor dust is constructed from the closed unit interval, which is denoted by [0, 1], by removing the open mid-third interval, denoted by ]1/3, 2/3[. This is followed by removing the open mid-thirds of [0, 1/3] and of [2/3, 1] and so on *ad infinitum*. The Cantor dust, defined as the set of all the points that are *never* removed, is a closed set. It is the sum of two halves, each deduced from the whole by a similarity of ratio $r = 1/b = 1/3$. Hence the fractal dimensionality of the dust is

$$D = \log N/\log(1/r) = \log 2/\log 3 \sim 0.6309.$$

The removed cut-outs are called 'tremas' (Mandelbrot 1982). Their endpoints obviously belong to the Cantor dust, but the dust also contains every point that is the limit of trema endpoints but is not itself a trema endpoint. The trema endpoints are denumerable (that is, can be labelled by an index whose values are the integers). Therefore, their fractal dimensionality is $D = 0$. (This $D$ cannot be obtained as $\log N/\log(1/r)$; a subtle issue is involved here; see Mandelbrot (1982, p 383).)

Now change Cantor's original procedure so that the tremas become closed intervals. In this case, the limit set can be described in intrinsic fashion as the difference of two sets having different fractal dimensionalities, namely the Cantor dust ($D \sim 0.6309$), minus the trema endpoints ($D = 0$). This difference is no longer a closed set.

The present paper involves analogous procedures performed in the plane and in higher spaces. Since a direct geometric analysis becomes unwieldy, we devised a general analytic approach. The transformations introduced in this fashion are linear, hence can be formulated using interscale transfer matrices, the TMFs. We show that the resulting fractals share the leading property of the 'Cantor dust minus its trema endpoints': in addition to the 'global' fractal dimensionality, these fractals involve the fractal dimensionalities of the parts that are either added or subtracted. We shall call them fractal 'eigen-dimensionalities'. We expect their values and relative weights to affect a system's physical character and expect our aglorithm to be relevant to physical problems.

Against the background of the one-dimensional 'Cantor dust minus the trema endpoints', the fact that our fractals decompose additively is not surprising. However, many interesting mathematical questions arise, some but not all of which we were able to solve. The decomposability of our fractal arises from the fact that the eigenvalues of the TMF are non-negative integers and its eigenvectors are real and satisfy some special orthogonality relations. These are very special properties for a matrix. Our algorithm is not yet completely understood from the mathematical viewpoint, and further understanding of its properties' relevance to physics is also needed. The purpose of this study is to stimulate efforts in both directions.

### *1.3. The occurrence of positive real eigenvalues in other problems of physics*

The fact that our TMFS have real and positive eigenvalues has counterparts in other problems of statistical physics. When the recursion relations of a renormalisation group can be presented as gradient flows, the eigenvalues $b^{\lambda_i}$ are real and positive, hence the exponents $\lambda_i$ of the eigenvalues are real (Wallace and Zia 1974, 1975). Gradient flows are found, to third order in $\varepsilon = 4 - d$, in the field-theoretical treatment of $n$-component continuous spin vector models. However, in the limit $n \to 0$, the flows cease to be described as gradient flows (Aharony 1975); some cases in this limit yield complex exponents. This limit corresponds to random systems, and it would be very interesting to know of similar results in other random systems.

## 2. Examples (non-random and random) and general rules

### *2.1. A basic example: original and depleted forms of the triadic Sierpiński carpet ($b = 3$, $l = 1$)*

We first recall the construction of the plane fractals called Sierpiński carpets, when they are viewed as collections of bonds. The general idea is to start with a quadratic lattice in the plane and to erase some of its bonds. Beginning with a square, one divides it into $b^2$ subsquares ($b$ is the base or the rescaling factor) and one declares that some of these subsquares form a 'trema' ( = hole). Thereafter, bonds are prohibited from crossing the trema, but they can belong to a trema's boundary. Usually, the number of subsquares in the trema is of the form $l^2$, with $l$ an integer. In each subsquare, a smaller trema is similarly positioned. In a first approximation, which will be discussed momentarily, the resulting fractal is self-similar and its fractal dimensionality, $D$, is given by $b^D = (b^2 - l^2)$.

Figure 1 shows two stages of construction of the original triadic carpet, for which $b = 3$, $l = 1$, and the trema is at the central subsquare. Here $D$ satisfies $3^D = 8$, hence $D \simeq 1.89$†.

Now, let us introduce and investigate a depleted form of the carpet. We start as in the original triadic construction, subdividing each square into $3^2$ subsquares. The novelty is that, when we eliminate the central subsquare, we also eliminate the bonds that bound it. Two steps of this construction are illustrated in figure 2.

The different shapes that are obtained after one step can be counted in several distinct ways. In a first step examined in this subsection, we do not take account of the different orientations, e.g. we consider the semi-open squares ⊔ and ⊏ as being the same shape. The first iteration yields four full squares and four squares that are open at one side (a, c, f, h, and b, d, e, g respectively, see figure 2$a$), yielding the

† The carpet is not quite self-similar because—just as in the first example in § 1.1—the eight subcarpets ovelap along intervals. The carpet becomes self-similar if, for example, one deletes the top and right sides of the original square, and the bottom and left sides of each trema. Thus, the carpet is more accurately described as the sum of a self-similar set ($D = 1.89$) and of a denumerable collection of lines ($D = 1$). However, taking this complication into account would make the intuitive treatment in § 2 needlessly cumbersome, and this complication is automatically avoided in the rigorous treatment of § 3.

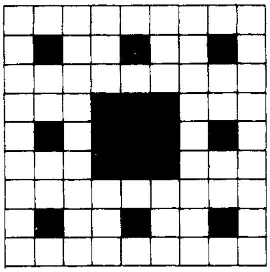
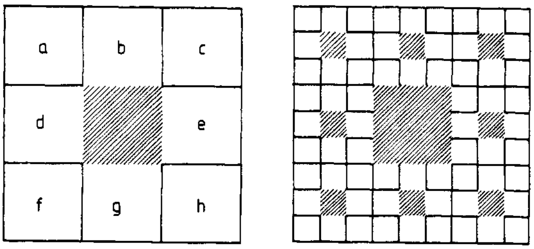


**Figure 1.** Two stages in the construction of Sierpiński carpet with $b=3$, $l=1$, $D=\log 8/\log 3 \simeq 1.89$.





($a$) ($b$)

**Figure 2.** Two consecutive stages in our construction, using $b=3$ and $l=1$ and starting from a 'full' square.

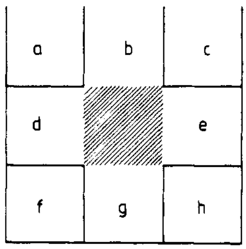




**Figure 3.** Subdivision of $\sqcup$-square (equation 2.2). Subsquares $f$, $h$ are full squares ($\square$). Subsquares a, c, d, e, g are denoted by ($\sqcup$) (we ignore difference in orientations) and b is denoted by ($[\,]$).

formal equation

$$\square \to 4\square + 4\sqcup. \tag{2.1}$$

The second iteration repeats the procedure. Its initial squares are either closed (as above) or open along one side. An analogous equation for this second type of squares is

$$\sqcup \to 2\square + 5\sqcup + 1[\,], \tag{2.2}$$

where $[\,]$ stands for a square which is open at two opposite sides. Equation (2.2) is illustrated geometrically in figure 3. Similarly,

$$[\,] \to 6\sqcup + 2[\,]. \tag{2.3}$$

It is essential that iterations beyond the second create no additional figures.

Being linear, the above scale transformation can be formulated in a three-dimensional vector space; for example, the vector $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$ stands for one full square $\square$, zero $\sqcup$ squares and zero $[\,]$ squares. Further, the transformation involves an interscale transfer matrix (TMF) operating on vectors on the right, namely

$$\begin{array}{c} \\ \square \\ \sqcup \\ [\,] \end{array} \begin{array}{c} \begin{array}{ccc} \square & \sqcup & [\,] \end{array} \\ \begin{pmatrix} 4 & 2 & 0 \\ 4 & 5 & 6 \\ 0 & 1 & 2 \end{pmatrix} \end{array}. \tag{2.4}$$

Operating $k$ times with the TMF on the initial vector yields a vector which has non-negative components, and describes the frequency of each shape after $k$ iterations. In the present example, the TMF has the eigenvalues $\lambda_1 = 8$, $\lambda_2 = 3$, $\lambda_3 = 0$, and the corresponding right eigenvectors (to be discussed below), are

$$\begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

### 2.2. *Triadic Sierpiński carpet with enlarged state space: first step*

We now enlarge the geometrical space on which the TMF operates. In the first step, we add three figures that cannot be created starting with the vector $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$. This yields a six-dimensional parameter space of shapes: □, ⊔, [ ], but also ⌟ (square with two neighbouring sides), _ (square with only one side left) and ⬚ (square with no side left). The new TMF takes the form

$$\begin{array}{c|cccccc} & □ & ⊔ & [\,] & ⌟ & \_ & ⬚ \\ \hline □ & 4 & 2 & 0 & 1 & 0 & 0 \\ ⊔ & 4 & 5 & 6 & 4 & 3 & 0 \\ [\,] & 0 & 1 & 2 & 2 & 3 & 4 \\ ⌟ & 0 & 0 & 0 & 1 & 2 & 4 \\ \_ & 0 & 0 & 0 & 0 & 0 & 0 \\ ⬚ & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \qquad (2.5)$$

where the order of the columns (as well as that of the rows) is self-explanatory. This TMF (Gefen 1983) has the eigenvalues $\lambda_1 = 8$, $\lambda_2 = 3$, $\lambda_3 = 1$, $\lambda_4 = \lambda_5 = \lambda_6 = 0$.

### 2.3. *Triadic Sierpiński carpet with an enlarged state space: second step*

Let us introduce a distinction between ⊔ and ⊐, and more generally between all the different possible orientations of the shapes considered in § 2.1. The resulting $16 \times 16$ TMF has the eigenvalues $\lambda_1 = 8$, $\lambda_2 = \ldots = \lambda_2 = 3$, $\lambda_6 = \ldots = \lambda_9 = 1$, $\lambda_{10} = \ldots = \lambda_{16} = 0$.

### 2.4. *General (non-triadic) Sierpiński carpets*

The case where $b - l = 2$ and the trema is positioned in the centre involves the same three-dimensional space as in equation (2.4). Unless otherwise stated, we do not distinguish between orientations. Hence, only the shapes that result from an initial □ are considered. The TMF is found to be

$$\begin{array}{c|ccc} & □ & ⊔ & [\,] \\ \hline □ & 4 & 2 & 0 \\ ⊔ & b^2 - l^2 - 4 & b^2 - l^2 - l - 2 & b^2 - l^2 - 2l \\ [\,] & 0 & l & 2l \end{array}, \qquad (2.6a)$$

that is

$$\begin{array}{c|ccc} & □ & ⊔ & [\,] \\ \hline □ & 4 & 2 & 0 \\ ⊔ & 4b - 4 & 3b - 4 & 2b \\ [\,] & 0 & b - 2 & 2b - 4 \end{array}. \qquad (2.6b)$$

This TMF has the eigenvalues $\lambda_1 = b^2 - l^2 = 4b - 4$, $\lambda_2 = b$, $\lambda_3 = 0$.

Changing the initial vector often necessitates an enlarged parameter space. Thus, if one starts from □ (and again $b-l=2$), the possible shapes are □, □, □ and □, and the TMF is

$$\begin{array}{c|c} & \begin{array}{cccc} \square & \square & \square & \square \end{array} \\ \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{pmatrix} 4 & 2 & 0 & 1 \\ 4b-4 & 3b-4 & 2b & 2b-2 \\ 0 & b-2 & 2b-4 & 2b-4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{2.7}$$

This TMF has the eigenvalues $\lambda_1=4b-4$, $\lambda_2=b$, $\lambda_3=1$, $\lambda_4=0$.

For the case $b-l>2$ (again the trema is in the centre), the TMF is the following $2\times 2$ matrix

$$\begin{array}{c|c} & \begin{array}{cc} \square & \square \end{array} \\ \begin{array}{c} \square \\ \square \end{array} & \begin{pmatrix} b^2-l^2-4l & b^2-l^2-4l-b \\ 4l & 4l+b \end{pmatrix} \end{array}. \tag{2.8}$$

The TMF has the eigenvalues $\lambda_1=b^2-l^2$, $\lambda_2=b$. Taking the orientations into account yields the TMF

$$\begin{array}{c|c} & \begin{array}{ccccc} \square & \square & \square & \square & \square \end{array} \\ \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \end{array} & \begin{pmatrix} b^2-l^2-4l & b^2-l^2-4l-b & b^2-l^2-4l-b & b^2-l^2-4l-b & b^2-l^2-4l-b \\ l & l+b & l & l & l \\ l & l & l+b & l & l \\ l & l & l & l+b & l \\ l & l & l & l & l-b \end{pmatrix} \end{array}. \tag{2.9}$$

This TMF has the eigenvalues $\lambda_1=b^2-\lambda^2$, $\lambda_2=\ldots=\lambda_5=b$.

### 2.5. *Low lacunarity lattices of base > 3*

For symmetric constructions with $l=(b-1)/2$, like that in figure 4, the TMF is

$$\begin{array}{c|c} & \begin{array}{ccc} \square & \square & \square \end{array} \\ \begin{array}{c} \square \\ \square \\ \square \end{array} & \begin{pmatrix} (b^2+2b+1)/4 & (b^2-1)/4 & (b^2-2b-3)/4 \\ 2b-2 & 2b-1 & 2b \\ (b^2-4b+3)/2 & (b^2-3b+2)/2 & (b^2-2b+1)/2 \end{pmatrix} \end{array}. \tag{2.10}$$

This TMF has the eigenvalues $\lambda_1=b^2-l^2=3b^2/4+\frac{1}{2}b-l/4$, $\lambda_2=b$ and $\lambda_3=0$.

### 2.6. *Sierpiński gasket*

In this interesting limit case, shown in figure 5, our algorithm continues to apply. The four-dimensional figure space contains the shapes △, △, △, △, and the TMF is

$$\begin{array}{c|c} & \begin{array}{cccc} \triangle & \triangle & \triangle & \triangle \end{array} \\ \begin{array}{c} \triangle \\ \triangle \\ \triangle \\ \triangle \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \end{array}. \tag{2.11}$$

This TMF has the eigenvalues $\lambda_1=3$, $\lambda_2=2$, $\lambda_3=1$, $\lambda_4=0$.

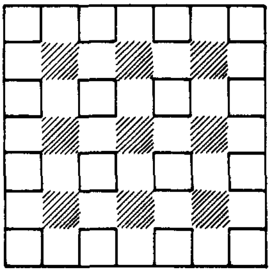


**Figure 4.** One construction stage for Sierpiński carpet with symmetrically smeared lacunas. Here $b=7$ and $l=4$.

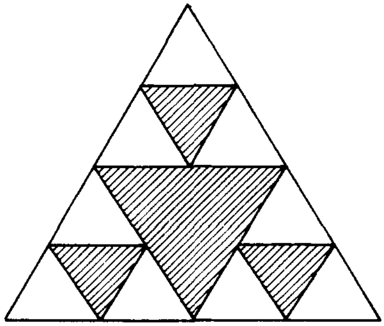


**Figure 5.** The Sierpiński gasket after two construction steps. $D=\log 3/\log 2 \simeq 1.59$.

Yet another possible construction focuses on the 'bonds' rather than the geometrical shapes. The Sierpiński gasket is geometrically equivalent to the Sierpiński carpet with $b=2$ and $l=1$ (figure 6). Here the tremas are not symmetrically located, and care should be taken to distinguish the different possible orientations of the squares. The resulitng $16\times 16$ TMF has the eigenvalues $\lambda_1=3$, $\lambda_2=\lambda_3=2$, $\lambda_4=\ldots=\lambda_8=1$, and $\lambda_9=\ldots=\lambda_{16}=0$.

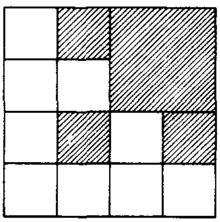


**Figure 6.** The Sierpiński carpet with $b=2$, $l=1$ after two construction steps. This structure is equivalent to the Sierpiński gasket (figure 5) with the same $D=\log 3/\log 2 \simeq 1.59$.

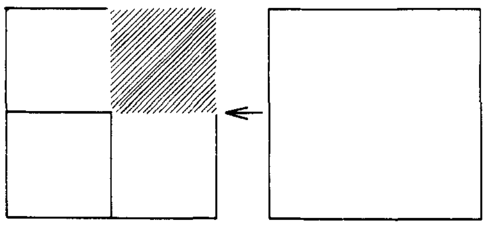


**Figure 7.** Modified construction of figure 6 with an initially full square, $b=2$, $l=1$. Internal bonds are eliminated when they lie on the boundary of the shaded area.

*2.6. Case where all the internal bonds are eliminated, except those adjacent to a trema*

Taking the last example ($b=2, l=1$) and eliminating all the internal bonds, except those adjacent to a trema, we find that an initially full square yields the following equation

$$\square \rightarrow 1\square + 1\square + 1\square. \tag{2.12}$$

Here again we have to consider separately the different possible orientations. The geometrical construction which corresponds to equation (2.12) is shown in figure 7. The resulting $16\times 16$ TMF has the eigenvalues $\lambda_1=3$, $\lambda_2=\lambda_3=2$, $\lambda_4=\ldots=\lambda_8=1$, and $\lambda_9=\ldots=\lambda_{16}=0$.

*2.7. Euclidean spaces with dimensionalities d higher than 2*

The algorithm in § 2.6 easily extends to $d>2$. Again one must decide how to distinguish between different shapes (according to their faces, edges, orientations, etc.). For example, consider the three-dimensional generalisation of the Sierpiński gasket (Mandelbrot 1982), constructed by connecting the midpoints of the edges of a tetrahedron, and eliminating the volume of the centre (bounded by the faces of the four new tetrahedra). One possible variant consists of eliminating all the faces that bound the eliminated central volume. The shape-space is now five-dimensional (the possible shapes are a 'full' tetrahedron, a tetrahedron with one missing face, two missing faces, etc). The TMF is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}. \tag{2.13}$$

This TMF has the eigenvalues $\lambda_1 = 4$, $\lambda_2 = 3$, $\lambda_3 = 2$, $\lambda_4 = 1$, $\lambda_5 = 0$. The $16 \times 16$ TMF, where different orientations are distinguished from each other, has the same eigenvalues as the TMF of (2.13), except that $\lambda_2$ and $\lambda_4$ are four-fold degenerate, and $\lambda_3$ is six-fold degenerate.

### *2.8. General properties of the TMF for the non-random algorithm: the eigenvalues*

The non-random constructions have been formulated via a TMF. The elements of this matrix are non-negative integers, but in general it is not symmetric. However, it turns out in every instance that the eigenvalues are non-negative real numbers; in general they are integers. For every TMF all the columns add to the same integer, $c$. Since the vector $(1, 1, \ldots, 1)$ is clearly a left eigenvector, $c$ turns out to be the largest non-degenerate eigenvalue; it is equal to $b^2 - l^2$ for the Sierpiński carpet. Each component of the right eigenvector associated with $c$ is positive, a consequence of the Perron–Frobenius theorem.

One should note that $\lambda_1 = c = b^2 - l^2 = b^D$, where $b$ is the rescaling factor, and $D$ is the global fractal dimensionality, which can be defined without using our algorithm. That is,

$$D = \log \lambda_1 / \log b. \tag{2.14}$$

But in addition, each smaller positive eigenvalue $\lambda_i$ injects a *fractal eigen-dimensionality*, defined as

$$D_i = \log \lambda_i / \log b.$$

The geometrical configuration we obtain involves intrinsically a number of distinct geometric objects with fractal dimensionalities smaller than $D$.

Eigenvalues equal to 0 play a particularly interesting role. Defining the minimal parameter space needed for a given construction we observe that the TMF may be either regular (e.g., (2.8)) or singular (e.g., (2.6)). *In the cases we studied, the TMF is singular when the order of ramification is finite.* We should recall that the order of ramification at a point $P$ is the number of points (bonds) that must be erased in order to isolate an arbitrarily large portion of the lattice containing $P$ (Mandelbrot 1982, chap 14). This notion is important in physics: for example, it was shown that for finitely ramified magnetic systems with finite range interactions, the critical temperature is zero (Gefen *et al* 1980, 1981a, 1983, 1984).

## *2.9. General properties of the* TMF *for the non-random algorithm: the eigenvectors*

The eigenvectors may also have some simple geometrical interpretation. Denote by $v_i (i = 1, 2, \ldots)$ the right eigenvectors corresponding to $\lambda_i (i = 1, 2, \ldots)$. Let the initial vector be expanded as $u = \Sigma a_i v_i$, with $a_i \neq 0$. After repeated iteration, the frequencies of the various geometrical shapes become proportional to the components of $v_1$. The fact that the other eigenvectors consist of both positive and negative components means that we cannot find an initial configuration that results in a single set with a fractal eigendimensionality smaller than $D$. Nevertheless, the existence of such sets (simultaneously with the set of global dimensionality $D$) cannot be denied. A simple example is the case described by the TMF in (2.7). Two stages of construction with an initial $\square$ square, $b = 3$ and $l = 1$ are shown in figure 8. The global fractal dimensionality of this figure is $D = \log 8/\log 3 = 1.89$. However, it is easy to see that this figure contains a subset converging towards the point A. For example, the fractal eigendimensionality $D_3 = \log \lambda_3/\log b = \log 1/\log 3 = 0$ reflects the presence of a set that reduces to such a subset.

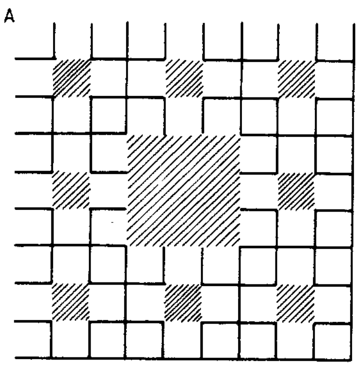



**Figure 8.** The construction (2.7), with an initial square after two steps. There is a subset converging towards the point A.

Other interesting properties were found by comparing 'families' of lattices that share the same $\lambda_1$, hence the same $D$, and the same shape space. An example consists of Sierpiński gaskets with $D = \log 32/\log 6$, as shown in figure 9. The resulting TMFs are listed below, with the corresponding eigenvalues and eigenvectors. We use the Dirac 'bra' and 'ket' notation, $\langle$ and $\rangle$, to denote non-normalised left and right eigenvectors. The different lattices are denoted by A, B, ..., E.

The TMF of A is

$$\begin{array}{cc} & \begin{array}{cccc} \square & \square & \square & \square \end{array} \\ \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{pmatrix} 24 & 18 & 12 & 13 \\ 8 & 14 & 20 & 18 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \qquad (2.16)$$

Here,

$$\lambda_1{}^{A} = 32, \qquad \lambda_2{}^{A} = 6, \qquad \lambda_3{}^{A} = 1, \qquad \lambda_4{}^{A} = 0,$$

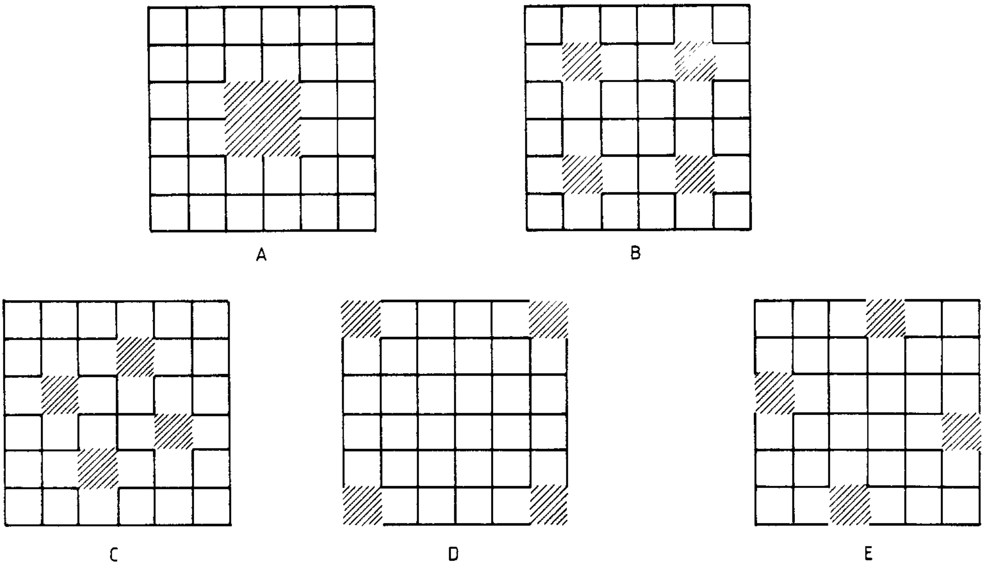



**Figure 9.** Five different fractals which belong to the same 'family'.

$$|A_1\rangle = \begin{pmatrix} 9 \\ 4 \\ 0 \\ 0 \end{pmatrix}, \qquad |A_2\rangle = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

$$|A_3\rangle = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \qquad |A_4\rangle = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix},$$

$$\langle a_1| = (1, 1, 1, 1), \qquad \langle a_2| = (4, -9, -22, -22),$$

$$\langle a_3| = (0, 0, 0, 1), \qquad \langle a_4| = (0, 0, 1, 0).$$

The TMF of B is

$$\begin{array}{c} \\ \square \\ \square \\ \square \\ \square \end{array} \begin{array}{c} \begin{array}{cccc} \square & \square & \square & \square \end{array} \\ \begin{pmatrix} 16 & 12 & 8 & 9 \\ 16 & 18 & 20 & 18 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{2.17}$$

Here,

$$\lambda_1{}^{B} = 32, \qquad \lambda_2{}^{B} = 6, \qquad \lambda_3{}^{B} = 1, \qquad \lambda_4{}^{B} = 0,$$

$$|B_1\rangle = \begin{pmatrix} 11 \\ 14 \\ 1 \\ 0 \end{pmatrix}, \qquad |B_2\rangle = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

$$|B_3\rangle = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \qquad |B_4\rangle = \begin{pmatrix} +1 \\ -2 \\ +1 \\ 0 \end{pmatrix},$$

$$\langle b_1| = (1, 1, 1, 1), \qquad \langle b_2| = (8, -5, -18, -18),$$

$$\langle b_3| = (0, 0, 0, 1), \qquad \langle b_4| = (1, -1, 3, -3).$$

The TMF of C is

$$\begin{array}{c} \\ \square \\ \square \\ \square \\ \square \end{array} \begin{array}{c} \begin{array}{cccc} \square & \square & \square & \square \end{array} \\ \begin{pmatrix} 16 & 11 & 6 & 7 \\ 16 & 20 & 24 & 22 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{2.18}$$

Here,

$$\lambda_1{}^{\mathrm{C}} = 32, \qquad \lambda_2{}^{\mathrm{C}} = 6, \qquad \lambda_3{}^{\mathrm{C}} = 1, \qquad \lambda_4{}^{\mathrm{C}} = 0,$$

$$|C_1\rangle = \begin{pmatrix} 21 \\ 30 \\ 1 \\ 0 \end{pmatrix}, \qquad |C_2\rangle = \begin{pmatrix} -5 \\ 4 \\ 1 \\ 0 \end{pmatrix},$$

$$|C_3\rangle = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \qquad |C_4\rangle = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix},$$

$$\langle c_1| = (1, 1, 1, 1), \qquad \langle c_2| = (8, -5, -18, -18),$$

$$\langle c_3| = (0, 0, 0, 1), \qquad \langle c_4| = (1, -1, 9, -3).$$

The TMF of D is

$$\begin{array}{c} \\ \square \\ \square \\ \square \\ \square \end{array} \begin{array}{c} \begin{array}{cccc} \square & \square & \square & \square \end{array} \\ \begin{pmatrix} 24 & 22 & 20 & 20 \\ 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 4 \end{pmatrix} \end{array}. \tag{2.19}$$

Here, $\lambda_1{}^{\mathrm{D}} = 32$, $\lambda_2{}^{\mathrm{D}} = 4$, $\lambda_3{}^{\mathrm{D}} = \lambda_4{}^{\mathrm{D}} = 0$, a possible choice of the eigenvectors is

$$|D_1\rangle = \begin{pmatrix} 41 \\ 14 \\ 0 \\ 1 \end{pmatrix}, \qquad |D_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad |D_3\rangle = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \qquad |D_4\rangle = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix},$$

$$\langle d_1| = (1, 1, 1, 1), \qquad \langle d_2| = (2, -5, -12, -12),$$

$$\langle d_3| = (1, -3, 8, 1), \qquad \langle d_4| = (1, -3, -7, 1).$$

The TMF of E is

$$\begin{pmatrix} 20 & 17 & 14 & 15 \\ 12 & 13 & 14 & 12 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 5 \end{pmatrix}. \tag{2.20}$$

Here

$$\lambda_1{}^{E}=32, \qquad \lambda_2{}^{E}=5, \qquad \lambda_3{}^{E}=1, \qquad \lambda_4{}^{E}=0,$$

$$|E_1\rangle=\begin{pmatrix} 163/4 \\ 27 \\ 0 \\ 2 \end{pmatrix}, \qquad |E_2\rangle=\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

$$|E_3\rangle=\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \qquad |E_4\rangle=\begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix},$$

$$\langle e_1|=(1,1,1,1), \qquad \langle e_2|=(4,-5,-14,-14),$$

$$\langle e_3|=(12,-19,-50,12), \qquad \langle e_4|=(0,0,1,0).$$

The above lattices A to E, and all the other examples we have considered, satisfy the following relation (derived by Yigal Meir)

$$\frac{\langle x_i\,|\,Y_j\rangle\langle y_j\,|\,X_i\rangle}{\langle x_i\,|\,X_i\rangle\langle y_j\,|\,Y_j\rangle}=\delta_{ij}. \tag{2.21}$$

Relation (2.21) is by no means a general property of stochastic matrices. It is a consequence of strong constraints imposed by our general geometrical constructions, as will be seen in § 3.

### 2.10. *Randomised forms of the same constructions*

One natural way to randomise our procedure is to choose the trema at each step at random, giving each subsquare (or subtriangle, etc.) equal probability, and then averaging over all possible TMFS. Working out the random Sierpiński gasket case we find the TMF to be

$$\begin{pmatrix} \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 2 & \frac{7}{4} & \frac{3}{4} \\ 0 & \frac{1}{2} & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

This TMF has the eigenvalues $\lambda_1=3$, $\lambda_2=3/2$, $\lambda_3=3/4$, $\lambda_4=0$. The equivalent random

Siepiński carpet with $b=2$ and $l=1$ yields

$$\begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 2 & \frac{3}{2} & 1 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{3}{4} & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}. \tag{2.23}$$

This TMF has the eigenvalues $\lambda_1=3$, $\lambda_2=3/2$, $\lambda_3=3/4$, $\lambda_4=\lambda_5=0$. The only change from the previous example is that the zero eigenvalue is doubly degenerate.

A more thorough randomisation is achieved by allowing the number of tremas at each step to vary, keeping only their average number fixed.

The above examples and many others suggest that many features of the TMFS of non-random constructions are left unchanged by randomisation. But there are important differences. In the random case, the components of the average TMF, while real and positive, are not necessarily integers. The sum of each column is still constant, hence the argument in § 2.8 shows that this constant is eigenvalue associated with the global fractal dimensionality. The corresponding eigenvector consists of positive components. However, the geometrical interpretation of this eigenvalue, as well as the definition of fractal eigendimensionalities, are not quite obvious, since it is not evident that they have to do with the average TMF.

The random TMFS obtained in this fashion correspond to fractal constructions where at each step different parts of the lattices may be decorated using different rules. Another situation occurs if at each construction step all parts of the lattice are decorated according to the same rule, but the rule at each step is selected at random. This corresponds to a random product of TMFS. It would be interesting to study such random products, and to identify the related fractal dimensionalities. When the TMFS are selected from the same family (in the sense discussed in § 2.9) we find that

$$\lambda_i^{XY}=\lambda_i^X\lambda_i^Y, \tag{2.24}$$

where $\lambda_i^{XY}$ is the ith eigenvalue of the product of matrices $X$ and $Y$. In this case the analysis of the dimensionalities involved is straightforward. We still need a better understanding of the last relation. For instance, one might ask whether it is connected with equation (2.21).

## 3. Rigorous mathematical treatment

### *3.1. Summary of the main results*

The main results of this section concern the eigenvalues of non-random enlarged TMFS, for which different orientations are regarded as different shapes. Although the discussion presented below is quite general, we begin with a specific example. Let us study the carpet constructed by iterating the shape shown in figure 10. One may now consider the set $S$ of the subsquares ($s=1,2,\ldots,9$) and two mappings, $\phi$ and $\tau$ of this set: $\phi(s)$ is the set of edges of $s$ which are contained in the boundary of the initial squares; $\tau(s)$ is the set of edges of $s$ which are not contained in the eliminated squares or in the boundary of the initial big square. Thus, e.g., $\phi(1)=\{\mathrm{n},\mathrm{w}\}$ and $\tau(1)=\{\mathrm{e}\}$.

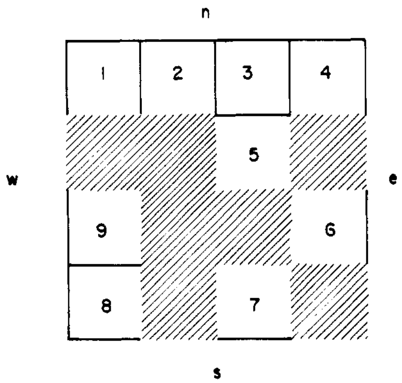



**Figure 10.** A general non-random carpet: see § 3.1.

Table 1 lists the other values of $\phi(s)$ and $\tau(s)$. In this example, each of the 16 eigenvalues of the TMF may be calculated as follows: we choose a combination of some elements of e, w, s, n, e.g., $I=\{n, w\}$ (notice that there are 16 possible combinations). We now count the number of squares $s$ such that $\phi(s)$ contains $I$. According to table 1 there is only $s=1$. The number of these terms, one, is an eigenvalue, $\lambda_I$, of the TMF. This suggests a geometrical interpretation of the eigenvalues: for $I=\varnothing$, $\lambda_I=9$, the number of subsquares, which is related to the primary fractal dimensionality. For $I=\{n\}$, $\lambda_I=4$, which corresponds to a partial fractal dimensionality of 1 (which, in this case, is the upper edge of the square in figure 10). $I=\{n, w\}$ yields $\lambda_1=1$ (a fractal eigen-dimensionality of zero) describing a subset which converges to the north-west point.

Other properties of the TMFs, as well as the orthogonality of the eigenvectors of 'families' of TMFs (equation 2.21) are also discussed.

**Table 1**

| $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\phi(s)$ | {n, w} | {n} | {n} | {n, e} | $\varnothing$ | {e} | {s} | {s, w} | {w} |
| $\tau(s)$ | {e} | {e, w} | {e, w, s} | {w} | {n} | $\varnothing$ | $\varnothing$ | {n} | {s} |

### *3.2. Notation*

*If* $E$ is a set, then $2^E$ stands for the set of subsets of $E$. If $I$ and $J$ are in $2^E$, then $I \Delta J$ is the subset of $E$, the elements of which belong to either $I$ or $J$ but not to both of them. The number of elements of a set $I$ is denoted by $\# I$.

### *3.3. Definitions*

Let $E$ be a finite set. An $E$-chessboard is a triplet $(S, \tau, \phi)$, where $S$ is a finite set and $\tau$ and $\phi$ are mappings from $S$ to $2^E$ such that, for every $s$ in $S$, $\tau(s) \cap \phi(s)=\varnothing$.

Two square matrices, $A$ and $M$, indexed by $2^E \times 2^E$, are associated to a chessboard:

$$A=\{a_J^I\}_{I,J\in 2^E}, \quad a_J^I=\#\{s \in S;\, \phi(s) \supset I \text{ and } \tau(s)=J\}, \tag{3.1}$$

$$M = \{m_J^I\}_{I,J \in 2^E}, \qquad m_J^I = \#\{s \in S;\ \tau(s) \cup (I \cap \phi(s)) = J\}. \tag{3.2}$$

It is clear that if $I \cap J \neq \emptyset$, then $a_J^I = 0$. It should be realised that $M$ is essentially a TMF.

If a square $s$ is in $S$, $\tau(s)$, the type of $s$ is the set of its edges, which are not the edge of a trema and which are not contained in the boundary of the initial square. The function $\phi$ describing the position of $s$ in the square is defined as follows: $\phi(s)$ is the set of the edges of $s$ which are contained in the boundary of the initial square of type $\emptyset$.

If, instead of starting from such a square, one starts from a square of type $I$, then the type of $s$ was to be modified: one should add the part of the boundary of $s$ contained in the boundary of the big square corresponding to $I$. In other words, as an offspring of a square of type $I$, a square $s$ has the type $\tau(s) \cup (\phi(s) \cap I)$. Thus $M$ is a TMF.

Generally speaking, TMFs are obtained from matrices $M$ by either grouping some types of squares or by taking a sub-matrix which corresponds to a recurrence class.

### *3.4. Expression of M in terms of the a's*

*Proposition.* If $(S, \tau, \phi)$ is an $E$-chessboard with matrices $A$ and $M$, then, for $I$ and $J$ in $2^E, m_J^I = \Sigma_{K \subset I} \Sigma_{L \subset K} (-1)^{|K|+|L|} a_{J\Delta L}^K$.

We prove it by induction. Let $H_\nu$ be the following property: for any chessboard the above formula is valid for an $I$ the cardinality of which is not larger than $\nu$.

It is obvious that $H_0$ is true. It has to be proven that $H_\nu$ implies $H_{\nu+1}$.

Assuming $H_\nu$ is true, $|I| = \nu$ and $i \in E \backslash I$, the following relation has to be shown:

$$m_J^{I \cup \{i\}} = \sum_{K \subset I} \sum_{L \subset K} (-1)^{|K|+|L|} a_{J\Delta L}^K + \sum_{K \subset I} \sum_{L \subset K} (-1)^{|K|+|L|+1} a_{J\Delta L}^{K \cup \{I\}}$$
$$+ \sum_{K \subset I} \sum_{L \subset K} (-1)^{|K|+|L|} a_{J\Delta(L \cup \{i\})}^{K \cup \{i\}}. \tag{3.3}$$

Two cases may occur:

(a) $i \notin J$. If so the following auxiliary $E'$-chessboard $(S', \tau', \phi')$ is of interest: $E' = E \backslash \{i\}$, $S' = \{s \in S;\ i \in \phi(s)\}$, $\tau'(s) = \tau(s)$ and $\phi'(s) = \phi(s) \backslash \{i\}$. It has matrices $A'$ and $M'$ and, if $K$ and $L$ are in $2^{E'}$,

$$a_L'^K = \#\{s \in S;\ \phi'(s) \supset K \text{ and } \tau'(s) = L\} = a_L^{K \cup \{i\}}$$

and

$$\begin{aligned} m_J^{I \cup \{i\}} &= \#\{s \in S;\ \tau(s) \cup [I \cup \{i\}) \cap \phi(s)] = J\} \\ &= \#\{s \in S;\ \tau(s) \cup (I \cap \phi(s)) = J\} \\ &\quad - \#\{s \in S';\ \tau(s) \cup (I \cap \phi(s)) = J\} \\ &= m_J^I - m_J'^I. \end{aligned}$$

Therefore, using $H_\nu$,

$$\begin{aligned} m_J^{I \cup \{i\}} &= \sum_{K \subset I} \sum_{L \subset K} (-1)^{K|+|L|} (a_{J\Delta L}^K - a_{J\Delta L}'^K) \\ &= \sum_{K \subset I} \sum_{L \subset K} (-1)^{|K|+|L|} a_{J\Delta L}^K - \sum_{K \subset I} \sum_{L \subset K} (-1)^{|K|+|L|} a_{J\Delta L}^{K \cup \{i\}}. \end{aligned}$$

But $a^{K \cup \{i\}} J\Delta(L \cup \{i\}) = 0$, therefore equation (3.3) is valid.

(b) $i \in J$. If so, a different auxiliary $E'$-chessboard $(S'', \tau'', \phi'')$ is used: $S'' = \{s \in S;\, i \in \tau(s)\}$, $\tau''(s) = \tau(s) \backslash \{i\}$, $\phi''(s) = \phi(s)$. It has matrices $A''$ and $M''$ and, if $K$ and $L$ are in $2^{E'}$,

$$a''^{K}_{L} = \#\{s \in S;\, \tau(s) \ni i,\, \tau(s)\backslash\{i\} = L \text{ and } \phi(s) \supset K\} = a^{K}_{L \cup \{i\}}$$

and

$$\begin{aligned} m_J^{I \cup \{i\}} &= \#\{s \in S';\, \tau(s) \cup [(I \cup \{i\}) \cap \phi(s)] = J\} \\ &\quad + \#\{s \in S;\, i \in \phi(s) \text{ and } \tau(s) \cup (I \cap \phi(s)) = J\} \\ &= \#\{s \in S';\, \tau'(s) \cup (I \cap \phi'(s)) = J\backslash\{i\}\} \\ &\quad + \#\{s \in S'';\, \tau''(s) \cup (I \cap \phi''(s)) = J\backslash\{i\}\}. \end{aligned}$$

Therefore, using $H_\nu$ and the expressions for $a'^{K}_{L}$ and $a''^{K}_{L}$,

$$\begin{aligned} m_J^{I \cup \{i\}} &= \sum_{K \subset I} \sum_{L \subset K} (-1)^{|K|+|L|} (a'^{K}_{(J\backslash\{i\})\Delta L} + a''^{K}_{(J\backslash\{i\})\Delta L}) \\ &= \sum_{K \subset I} \sum_{L \subset K} (-1)^{|K|+|L|} (a^{K \cup \{i\}}_{(J\backslash\{i\})\Delta L} + a^{K}_{[(J\backslash\{i\})\Delta L] \cup \{i\}}) \\ &= \sum_{K \subset I} \sum_{L \subset K} (-1)^{|K|+|L|} a^{K \cup \{i\}}_{J \Delta (L \cup \{i\})} + \sum_{K \subset I} \sum_{L \subset K} (-1)^{|K|+|L|} a^{K}_{J \Delta L}. \end{aligned}$$

But, if $L \subset I$, $a^{K \cup \{i\}}_{J \Delta L} = 0$, hence equation (3.3) is valid again.

### *3.5. Sum of chessboards*

Let $(S', \phi', \tau')$ and $(S'', \phi'', \tau'')$ be two $E$-chessboards. Let $s$ be the disjoint union of $S'$ and $S''$ and $\phi$ and $\tau$ the mappings from $S$ to $2^E$, the restrictions of which to $S'$ and $S''$ are $\phi'$, $\phi''$, $\tau'$ and $\tau''$. Then $(S, \phi, \tau)$ is also an $E$-chessboard and their matrices are related as following: $M = M' + M''$ and $A = A' + A''$.

### *3.6. Product of chessboards*

Let $(S', \phi', \tau')$ and $(S'', \phi'', \tau'')$ be two $E$-chessboards. Let $S$ be the cartesian product $S' \times S''$ and define

$$\phi(s', s'') = \phi'(s') \cap \phi''(s'')$$

and

$$\tau(s', s'') = \tau''(s'') \cup (\tau'(s') \cap \phi''(s'')).$$

Then $(S, \phi, \tau)$ is an $E$-chessboard. One has

$$\begin{aligned} &\{(s', s'') \in S;\, \tau''(s'') \cup (\tau'(s') \cap \phi''(s'')) \cup (I \cap \phi'(s') \cap \phi''(s'')) = J\} \\ &\quad = \bigcup_{K \in 2^E} \{(s', s'') \in S;\, \tau''(s'') \cup (\tau'(s') \cap \phi''(s'')) \cup (I \cap \phi'(s') \cap \phi''(s'')) = J\} \\ &\qquad \cap \{(s', s'') \in S;\, \tau'(s') \cup (I \cap \phi'(s')) = K\} \\ &\quad = \bigcup_{K \in 2^E} \{(s', s'') \in S;\, \tau''(s'') \cup (K \cap \phi''(s'')) = J\} \end{aligned}$$

$$\cap\{(s', s'')\in S;\ \tau'(s')\cup(I\cap\phi'(s'))=K\}$$

$$=\bigcup_{K\in 2^E}\{s''\in S'';\ \tau''(s'')\cup(K\cap\phi''(s''))=J\}$$

$$\times\{s'\in S';\ \tau'(s')\cup(I\cap\phi'(s'))=K\}.$$

So, with obvious notation,

$$m_J^I=\sum_{K\in 2^E} m_J''^K m_K'^I,$$

therefore $M=M''M'$.

### 3.7. Spectral study of TMFs

*Theorem 1.* Let $M$ be a TMF associated to an $E$-chessboard $(S, \phi, \tau)$. Then the eigenvalues of $M$ are the numbers

$$\lambda_I=\#\{s\in S;\ \phi(s)\supset I\}=\sum_{J\in 2^E} a_J^I \qquad (I\in 2^E).$$

*Theorem 2.* If $M'$ and $M''$ are TMFs associated to $E$-chessboards, then the eigenvalues of $M=M''M'$, $M'$ and $M''$ are related by:

$$\lambda_I=\lambda_I'\lambda_I'' \qquad (I\in 2^E).$$

The proof of the first assertion is carried out by induction on $|E|$. Let $i$ be an element of $E$. If $I$ and $J$ are in $2^{E\setminus\{i\}}$, then, by the preceding proposition,

$$m_J^{I\cup\{i\}}=m_J^I+\sum_{K\subset I}\sum_{L\subset K}(-1)^{|K|+|L|+1}a_{J\Delta L}^{K\cup\{i\}}$$

and

$$m_{J\cup\{i\}}^{I\cup\{i\}}=m_{J\cup\{i\}}^I+\sum_{K\subset I}\sum_{L\subset K}(-1)^{|K|+|L|+2}a_{J\Delta L}^{K\cup\{i\}}.$$

Therefore the matrix $M$ assumes the form:

$$M=\begin{pmatrix} M_1 & M_1-M' \\ M'' & M''+M' \end{pmatrix}.$$

where $m_{1J}^I=m_J^I$, $m_J''^I=m_{J\cup\{i\}}^I$ and

$$m_J'^I=\sum_{K\subset I}\sum_{L\subset K}(-1)^{|K|+|L|}a_{J\Delta L}^{K\cup\{i\}}.$$

The matrices $M_1$, $M'$ and $M''$ are the TMFs of $E\setminus\{i\}$-chessboards. Namely, $M_1$ is the matrix of the $E\setminus\{i\}$-chessboard $(S_1, \phi_1, \tau_1)$ defined as follows: $S_1=\{s\in S,\ i\notin\tau(s)\}$, $\tau_1=\tau|_{S_1}$, $\phi_1(s)=\phi(s)\cap(E\setminus\{i\})$. $M'$ and $M''$ are the matrices of the chessboards $(S', \phi', \tau')$ and $(S'', \phi'', \tau'')$ which have already been defined in § 3.4.

It is now easy to determine the eigenvalues of $M$:

$$\det(M-\lambda)=\det\begin{pmatrix} M_1-\lambda & M_1-M' \\ M'' & M''+M'-\lambda \end{pmatrix}$$

$$=\det\begin{pmatrix} M_1+M''-\lambda & M_1+M''-\lambda \\ M'' & M''+M'-\lambda \end{pmatrix}$$

$$= \det \begin{pmatrix} M_1 + M_1'' - \lambda & 0 \\ M'' & M' - \lambda \end{pmatrix}.$$

By the induction hypothesis, the eigenvalues of $M'$ are the numbers $\{\Sigma_{J \subset E \setminus \{i\}} (a_J^{I \cup \{i\}}\}_{I \subset E \setminus \{i\}}$ and those of $M_1 + M''$ are $\{\Sigma_{J \subset E \setminus \{i\}} (a_J^I + a_{J \cup \{i\}}^I)\}_{I \subset E \setminus \{i\}}$ and the first assertion of the theorem is proved.

The second assertion results from the formula $\lambda_I = \#\{s \in S;\ \phi(s) \supset I\}$ and from the definition of the product of two chessboards.

*3.8. Remark on the orthogonality relations equation (2.21)*

Let us consider two TMFS, $M$ and $N$, associated to $E$-chessboards. Assuming $M$ and $N$ are diagonalisable, one can find right and left eigenvectors for $M$ and $N$, so that the relation (2.21) holds. This is proved by employing the decomposition

$$M = \begin{pmatrix} M_1 & M_1 - M' \\ M'' & M'' + M' \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} N_1 & N_1 - N' \\ N'' & N'' + N' \end{pmatrix} \quad \text{considered above.}$$

## 4. Discussion and directions for further study

The present paper presents an algorithm of constructing geometrical shapes with which a multiplicity of non-integer fractal dimensionalities can be associated. It indicates some of the general features of the TMF related to both the non-random and random construction (§§ 2.8 and 3 respectively). The richness and the generality of our algorithm allows for further extensions and modifications of such constructions. We emphasise that a deep and rigorous mathematical understanding of the striking properties of the TMF is still lacking. Further study of the fractal eigen-dimensionalities and the geometrical interpretation of the eigenvectors associated with them, as well as additional effort in understanding the singularities of the TMFS are needed. The situation in the random cases is even less obvious, and in addition to the preceding points one should consider how an appropriate averaging in the random case should be performed. We hope that our preliminary study will stimulate such efforts.

Finally, we would like to comment on the possible application of the method described here to physical problems. It is tempting to use this approach to study self-similar structures that appear in nature. Results on the analysis of percolation clusters using TMFS will be published elsewhere (naturally, in that case randomness plays an important role). We hope that our work will stimulate studies of other problems, including kinetic ones (e.g., structures of diffusion limited aggregation) by the means furnished here.

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